

TEACHERS KNOWLEDGE OF INFINITY, AND ITS ROLE IN CLASSROOM PRACTICE

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This paper considers infinity as an element of professional knowledge. We assume that teachers need a wider knowledge of the topic than that they possess as mathematics students. Using three models of professional knowledge, Mathematics Teacher's Specialised Knowledge (MTSK), Mathematical Knowledge for Teaching (MKT), and Knowledge Quartet (KQ), we discuss how the notions being proposed in these models, and on which they are constructed, might contribute to studying teachers' knowledge of infinity.

INTRODUCTION

Infinity as a learning item has been widely studied, from the seminal work of Fischbein, Tirosh and Hess (1979), to more recent contributions (e.g. Zoitsakos, Zachariades and Sakonidis, 2013; Dubinsky, Arnon, Weller, 2013). Worthy of note in this respect is the bibliographic review by Belmonte (2009), which considers over 300 published papers on the topic. The focus of these publications concerns mainly two aspects: the process of developing the cognition of infinity (e.g. Lakoff & Nuñez, 2001), and differing conceptions of infinity (e.g. Belmonte 2009). Recently, the research community has begun to show interest in the understanding that prospective teachers have of infinity (e.g. Manfreda Kolar & Hodnik Cadez 2012, Dubinsky et al. 2013). Most of the research on these two aspects focuses on students or prospective teachers, leaving aside practising teachers and their knowledge, as well as their role in and for practice. Although we concur with the philosophy underlying previous research, and recognize the value of understanding the degree of teachers' cognitive development with respect to infinity, we feel an approach to teachers' knowledge of infinity from the perspective of professional knowledge should not be limited to an "*on its head*" knowledge (Thames and VanZoest, 2013, p. 592), but should rather consider in what way the teacher understands infinity in the teaching and learning context, and how he or she uses (or can use) this knowledge in their professional practice. This leads to the question of whether a teacher should understand infinity differently to the pupil (and if so, how), or should simply understand it to a more advanced degree. This paper draws together perspectives on this question, from both the literature on learning about infinity and the field of professional knowledge, with the aim of providing insights for future research into teachers' knowledge of the topic.

INFINITY IN THE CLASSROOM

Infinity as a mathematical item is not typically explicit on primary and secondary syllabuses (e.g. NCTM, 2000), although it is to be found as a backdrop to certain mathematical notions, such as in the concept of limit or the measurement of area using integrals, and in others underlying basic processes, such as counting processes and number systems generation (Gardiner, 1985), both of which do have a place on the syllabus in various countries. A broad overview of curricular content leads us to wonder whether teacher training should contemplate the inclusion of mathematical aspects (the epistemological and phenomenological, amongst others) as well as didactic considerations, so that teachers know and understand infinity when it comes up in the curriculum, and, more significantly, are able to recognise it as a latent presence underlying a number of mathematical topics.

There are many approaches to infinity which take the pupils' point of view into account and tackle the topic intuitively. Many centre on the different types of reasoning called upon to understand iterative processes, or on pupils' own 'naturally' expressed definitions when dealing with concepts such as limit, density, and the periodicity of the decimal part of a number.

Recently, reviewing the notions put forward by different authors, Belmonte (2009) detected six different intuitive patterns underlying secondary pupils' understanding of infinity, for which he aimed to group the different classification systems deployed in previous research within a single system, including several novel notions about finding the sum of a series. Such studies are of even greater interest when considered alongside research into how topics such as limit are explored in class (e.g. Sierpiska, 1987), as direct classroom applications become apparent, with examples from real lessons involving discussions with pupils in which they reflect and articulate their own understanding of infinity.

From the point of view of teacher knowledge and training, it is not unreasonable to think that teachers should have a good working knowledge of the stages pupils need to pass through to achieve an understanding of infinity, as this will enable them to respond appropriately to their pupils, and to better select, organise and sequence classroom tasks. However, we would argue that in addition to understanding these developmental aspects of infinity; teachers should also know how to introduce the concept to their class in such a way as not to limit their pupils' development. Likewise, they should be aware of how certain conceptualisations of infinity limit the mathematical constructs that can be built, as shall be seen below.

AN EXAMPLE

The extracts presented below are taken from a discussion between the first author (R) and a secondary teacher (A) about an example given in Belmonte (2009). The three excerpts appear in chronological order and correspond to three different points in a continuous discussion:

Example under discussion:

Everyone imagine a number. Halve it. Halve the results, and so on successively. What is the final result?

A pupil answers:

We don't know, because we don't know when to stop.

(Belmonte, 2009)

Extract 1

- A: This is like the example of the jumping frog, which jumps towards the edge of the reservoir. I used to use it, but not any more.
- R: Why not?
- A: Because there were arguments. [...] One person wouldn't accept it, while another would, and in the end they'd get angry and would say, "Well, I don't" and the other would say, "Well, I do," and they just wouldn't agree.
- R: And why did one accept it and the other didn't?
- A: Because of the physical aspect. You explain the sequences to them, how it works – half the length of the previous jump, then half again, and half again . . . and a lot of them say that the frog makes it. Others say when it gets close, it takes a bigger jump and gets there.

Extract 2

- A: Sometimes they're given an example like the other day, the frog that jumps halfway. Does it reach, or not?
- R: OK, and does it reach, or not?
- A: No, no, it doesn't.
- R: OK.
- A: It does reach the limit, but it doesn't. In physical terms, it shouldn't reach it.

Extract 3

- A: I'll give you the definition I give to my pupils. Infinity is something invented to explain the inexplicable. [. . .] Unknown, untouchable. Not invented, but it's there to explain something which doesn't have an explanation really.

We will use the teacher's statements to analyse aspects of the conceptualisation of infinity using notions drawn from various models of professional knowledge, specifically Knowledge Quartet (Rowland, Turner, Thwaites & Huckstep, 2009), Mathematical Knowledge for Teaching (Ball, Thames & Phelps 2008) and Mathematics Teacher Specialized Knowledge (Carrillo, Climent, Contreras & Muñoz-Catalán, 2013). We will organise the analysis in terms of the domains of

Mathematical Knowledge and Pedagogical Content Knowledge (PCK), derived from Shulman (1986), for their compatibility with the above models.

ANALYSIS

Mathematical knowledge

The teacher displays understanding of certain phenomenological aspects of infinity, such as the concept of limit, in that he expounds upon an example demonstrating an underlying notion of infinity as a gradual approach to a limit (from a clearly potential perspective), based on one of Zeno's paradoxes. Additionally, not only is he capable of establishing the connection between the limit and the example, but also, from his way of conceptualising infinity, he is able to discuss the example. This "*dealing with infinity*" is one of the Big Ideas (Kuntze, Lerman, Murphy, Kurz-Milcke, Siller & Windbourne, 2011) in relation to mathematical content. Considered in terms of KQ, it can be seen as pertaining to *Foundations* (Rowland et al., 2009), as it constitutes the theoretical background to various ideas, while at the same time forming part of *Connections* (ibid.), in that it puts the teacher's mathematical connections into action. Seen through the lens of MTSK, the Big Idea comes within the scope of *Knowledge of the Structure of Mathematics* (Carrillo et al., 2013), as it cuts across mathematical categories, and could be regarded as a foundation stone of school mathematics, lending theoretical support to a multitude of concepts. Regarding MKT, it is possible to argue for its inclusion in different subdomains. It seems clear that infinity cannot be regarded as pertaining to *Common Content Knowledge*, as it is beyond what might reasonably be expected of someone with mathematical schooling (given that, as mentioned above, there is not usually any specific focus on it), and as such it seems more appropriate, as an item exclusive to teaching, to assign it to the domain of *Specialised Content Knowledge*. In like fashion, it can be argued that an understanding of infinity, and the way in which this organises other mathematical concepts, fulfil the criteria for what Jakobsen, Thames and Ribeiro (2013) denominate "Familiarity with the discipline", as a characteristic of *Horizon Content Knowledge*.

In the case of our teacher, his conceptualisation of infinity as something unknown and artificial leads him to affirm that, although the concept of limit exists, the idea of its 'reachability' would not make sense in real life, illustrating a certain confusion between context and problem. As a result of his interpretation of infinity, the teacher fails to abstract the situation to a mathematical context. This process of modelling, which requires the teacher to be aware of the need to do so (for example, in terms of his objectives in employing a particular example), leads us to another component of mathematical knowledge, Knowledge of the Practice of Mathematics, (Carrillo et al, 2013), within the perspective of MTSK, or the Horizon Content Knowledge associated with the practice of mathematics (Ball and Bass, 2009). In this respect, given that the understanding involved is close to syntactic, we can understand the use of the notion of Foundations (Rowland et al. 2009).

Pedagogical content knowledge

Seen through the lens of PCK, defined in Shulman's (1986) seminal work as the knowledge which includes "ways of representing and formulating the subject that make it comprehensible to others" (ibid. p. 9), we note that the teacher chose a specific example to tackle a particular content associated with infinity, in this case the limit of a sequence. This choice, and the knowledge of the example itself as a means of representing the content, leads us to the need to consider PCK as applicable to infinity. In this case, the use of the three models above allows us to observe the teacher's knowledge from a very similar standpoint. Consistent with the observations made above, *Transformation* is present, this element of Knowledge Quartet being very close to Shulman's original definition of PCK. In the cases of MKT and MTSK, both models accommodate subdomains encompassing the choice of powerful examples for a particular content, *Knowledge of Content and Teaching*, in the case of MKT, and *Knowledge of Mathematics Teaching* in that of MTSK.

In MKT and MTSK, PCK is explicitly divided into three different subdomains, one for teaching mentioned above, another for the curriculum (in MKT, identical to *Curricular Knowledge* proposed by Shulman, 1986) or learning standards (representing an amplification in MTSK of the earlier work), and a final subdomain focusing on the students (MKT), or the characteristics of learning related to mathematics (MTSK). This kind of knowledge is visible in the case of the example above, in that the teacher is able to predict a typical answer, "it [the frog] makes a bigger jump and gets there," thus illustrating his understanding that some pupils are prevented from conceptualising the infinite reiteration of the process by the barrier which the context represents for them.

FINAL REFLECTIONS

Infinity is an item intrinsic to school mathematics, frequently non-explicit, requiring an approach beyond consideration of the process by which it is learnt, as has largely been the case to date. This paper represents a call to tackle the concept as an item of professional knowledge, applicable to the day-to-day work of teaching, while taking into account the cognitive aspects affecting the teacher's understanding of infinity (as a learner). The different frameworks that have been applied support this notion. Each, incorporating its own theoretical constructs, helps us to better understand the conceptualisation of infinity brought into play by mathematics teachers tackling the various topics which constitute the phenomenology of the concept. The notion of structural concept in mathematics, which derives from the MTSK model is of special interest for us, along with the consideration that knowledge of infinity is exclusive to teachers, and pertains to the specialised content knowledge subdomain of MKT. With respect to KQ, *Transformation* represents a powerful means by which to consider the pedagogical implications of infinity.

We recognise that the field under consideration, teachers' knowledge of infinity, is a recent innovation. We hope that these considerations are followed by others which

enable them to be amplified. In the long term, we regard the inclusion of aspects of infinity in teacher training programmes as one of the challenges facing this field.

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